A Majorization Relation Between the Height and the Level Characteristics*

Daniel Hershkowitz
Mathematics Department
Technion — Israel Institute of Technology
Haifa 32000, Israel

Submitted by Richard A. Brualdi

ABSTRACT

A recently proven graph theoretic lower bound for the nullity of powers of matrices in a triangular block form is used to obtain a majorization relation between the height and the level characteristics of a general matrix. An application of the result to the M-matrix case yields an improvement of a previously known result.

1. INTRODUCTION

The relation of the (spectral) height characteristic of a square matrix (associated with the eigenvalue 0) and its (graph theoretic) level characteristic has been the theme of several mathematical investigations, in particular for M-matrices; see e.g. Hershkowitz and Schneider [2] and the references there.

In this paper we use a graph theoretic lower bound for the nullity of powers of matrices in a triangular block form, recently proven by Friedland and Hershkowitz [1], in order to obtain a majorization relation between the height and the level characteristics of a general matrix. An application of our theorem to the M-matrix case yields a result that improves a previous result due to Richman and Schneider [3]. While the result in [3] asserts that the

*The research was carried out while the author was a visiting professor at the University of Wisconsin—Madison.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

height characteristic majorizes the level characteristic, we prove that the height characteristic majorizes every permutation of the level characteristic.

Section 2 contains the definitions and notation used in this paper. The results are contained in Section 3.

2. NOTATION AND DEFINITIONS

In this paper we assume that A is a complex $n \times n$ matrix, partitioned in a lower triangular $r \times r$ block form $(A_{ij})_1^r$, with square diagonal blocks. The index of A, that is, the size of the largest Jordan block associated with 0 as an eigenvalue of A, is assumed to be p.

DEFINITION 2.1. The *reduced graph* $R(A)$ of A is defined to be the directed graph with vertex set $\{1,\dots,r\}$, and such that there is an arc from i to j if $i \neq j$ and $A_{ij} \neq 0$.

Note that since A is a lower triangular block matrix, $R(A)$ contains no cycle.

DEFINITION 2.2. A vertex i in $R(A)$ is said to be *singular* if A_{ii} is singular. The set of all singular vertices of $R(A)$ is denoted by S.

DEFINITION 2.3. The *singular graph* $\mathrm{S}(A)$ of A is defined to be the directed graph with vertex set S, and such that there is an arc from i to j if $i = j$ or if there is a path from i to j in $R(A)$. Note that $\mathrm{S}(A)$ is a transitive graph with no cycle.

DEFINITION 2.4. Let $i \in \mathrm{S}$. We define the *level* of i as the maximal length (number of vertices) of a simple path in $\mathrm{S}(A)$ that terminates at i. We call the set of all vertices of level j the jth level of $\mathrm{S}(A)$, and we denote it by L_j. The cardinality of L_j is denoted by $\lambda_j(A)$. Let $\mathrm{S}(A)$ have q levels. The sequence $(\lambda_1(A),\dots,\lambda_q(A))$ is called the *level characteristic* of A, and is denoted by $\lambda(A)$. Normally we write λ_i for $\lambda_i(A)$ where no confusion should result.

NOTATION 2.5. For a square matrix B we denote by $n(B)$ the nullity of B (the dimension of the nullspace of B).

DEFINITION 2.6. For $i \in \{1,\dots,p\}$ let $\eta_i(A) = n(A^i) - n(A^{i-1})$. The sequence $(\eta_1(A),\dots,\eta_p(A))$ is called the *height characteristic* of A, and is denoted by $\eta(A)$. Normally we write η_i for $\eta_i(A)$ where no confusion should result.

We remark that the height characteristic of A is often referred to as the *Weyr characteristic* of A.

Convention 2.7. The level characteristic of A will be assumed to be $(\lambda_1,\dots,\lambda_q)$. The height characteristic of A will be assumed to be $(\eta_1,\dots,\eta_p)$. Whenever $i > p$ we assume that $\eta_i = 0$.

3. RESULTS

Let $\mathscr{k} = (k_1,\dots,k_r)$ be a sequence of nonnegative integers. For a path $\gamma = (i_1,\dots,i_s)$ in $S(A)$ we define

$$k(\gamma) = k_{i_1} + k_{i_2} + \cdots + k_{i_s}.$$

We then define

$$m(A,\mathscr{k}) = \max\{k(\gamma) : \gamma \text{ is a path in } S(A)\}.$$

The main theorem in [1] can be stated as

THEOREM 3.1. *Let $t = m(A,\mathscr{k})$. Then*

$$n(A^t) \geqslant \sum_{i \in S} n(A_{ii}^{k_i}).$$

Theorem 3.1 yields the following.

THEOREM 3.2. *Let T be a subset of $\{1,\dots,q\}$ of cardinality t. Then*

$$n(A^t) \geqslant \sum_{j \in T} \sum_{i \in L_j} n(A_{ii}). \tag{3.3}$$

Proof. For $i \in S$ we choose $k_i = 1$ if $i \in L_j$ for some $j \in T$, and $k_i = 0$ otherwise. Also, for $i \notin S$ we set $k_i = 0$. It follows from Definition 2.4 that no path in $S(A)$ contains two vertices of the same level. Therefore, since the cardinality of T is t, it now follows that every path in $S(A)$ contains at most t vertices with levels in T. Also, at least one path in $S(A)$ contains exactly t vertices with levels in T. It now follows that the sequence $\ell = (k_1,\dots,k_r)$ satisfies $m(A,\ell) = t$, and (3.3) follows from Theorem 3.1. ∎

COROLLARY 3.4. *Let T be a subset of $\{1,\dots,q\}$ of cardinality t. Then*

$$\sum_{i=1}^{t} \eta_i \geqslant \sum_{j \in T} \lambda_j.$$

Proof. Note that $n(A^t) = \eta_1 + \cdots + \eta_t$ and that the cardinality of L_j is λ_j. Since all the vertices i on the right hand side of (3.3) are singular, we have $n(A_{ii}) \geqslant 1$ for each of them. Therefore, Theorem 3.2 immediately implies the corollary. ∎

We define $(\hat{1},\dots,\hat{q})$ to be a permutation of $(1,\dots,p)$ such that $\lambda_{\hat{1}} \geqslant \cdots \geqslant \lambda_{\hat{q}}$. It now follows from Corollary 3.4 that

THEOREM 3.5. *Let $t \in \{1,\dots,q\}$. We have*

$$\sum_{i=1}^{t} \eta_i \geqslant \sum_{i=1}^{t} \lambda_{\hat{i}}.$$

If for every singular vertex i, 0 is a simple eigenvalue of A_{ii}, then we have

$$\sum_{i=1}^{p} \eta_i = \sum_{i=1}^{q} \lambda_i, \tag{3.6}$$

since both sides of (3.6) equal the algebraic multiplicity of 0 as an eigenvalue of A. A special such case is of M-matrices. Recall that A is said to be an *M-matrix* if it can be written as $aI - B$ where B is a nonnegative matrix, and where a is greater than or equal to the spectral radius of B. If A is an M-matrix, then we also have $p = q$; see [4]. Therefore, Theorem 3.5 yields the following theorem for M-matrices. This result improves Corollary 4.5 of [3] (see also Corollary (4.21) in [2]). While the result in [3] asserts that the height characteristic majorizes the level characteristic, we claim that the height characteristic majorizes every permutation of the level characteristic.

THEOREM 3.7. *Let A be an M-matrix, and let* $t \in \{1,\dots,p\}$. *Then*

$$\sum_{i=1}^{t} \eta_i \geqslant \sum_{i=1}^{t} \lambda_{\hat{i}},$$

with equality for $t = p$.

We conclude with a corollary that follows immediately from Theorem 3.7.

COROLLARY 3.8. *Let A be an M-matrix. Then* $\eta_p \leqslant \lambda_{\hat{p}}$.

REFERENCES

1 S. Friedland and D. Hershkowitz, The rank of powers of matrices in a block triangular form, *Linear Algebra Appl.* 107:17–22 (1988).
2 D. Hershkowitz and H. Schneider, Height bases, level bases, and the equality of the height and the level characteristics of an *M*-matrix, to appear.
3 D. J. Richman and H. Schneider, On the singular graph and the Weyr characteristic of an *M*-matrix, *Aequationes Math.* 17:208–234 (1978).
4 U. G. Rothblum, Algebraic eigenspaces of non-negative matrices, *Linear Algebra Appl.* 12:281–292 (1975).

Received 28 November 1988; final manuscript accepted 1 December 1988